

ANALYTIC CONTINUATION OF WEIGHTED q -GENOCCHI NUMBERS AND POLYNOMIALS

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ABSTRACT. In the present paper, we analyse analytic continuation of weighted q -Genocchi numbers and polynomials. A novel formula for weighted q -Genocchi-Zeta function $\tilde{\zeta}_{G,q}(s | \alpha)$ in terms of nested series of $\tilde{\zeta}_{G,q}(n | \alpha)$ is derived. Moreover, we introduce a novel concept of dynamics of the zeros of analytically continued weighted q -Genocchi polynomials.

1. INTRODUCTION

In this paper, we use notations like \mathbb{N} , \mathbb{R} and \mathbb{C} , where \mathbb{N} denotes the set of natural numbers, \mathbb{R} denotes the field of real numbers and \mathbb{C} also denotes the set of complex numbers. When one talks of q -extension, q is variously considered as an indeterminate, a complex number or a p -adic number.

Throughout this work, we will assume that $q \in \mathbb{C}$ with $|q| < 1$. The q -integer symbol $[x : q]$ denotes as

$$[x : q] = \frac{q^x - 1}{q - 1}.$$

Firstly, analytic continuation of q -Euler numbers and polynomials was investigated by Kim in [1]. He gave a new concept of dynamics of the zeros of analytically continued q -Euler polynomials. Actually, we were motivated from his excellent paper which is "Analytic continuation of q -Euler numbers and polynomials, Applied Mathematics Letters 21 (2008) 1320-1323." We also procure to analytic continuation of weighted q -Genocchi numbers and polynomials as parallel to his article. Also, we give some interesting identities by using generating function of weighted q -Genocchi polynomials.

2. PROPERTIES OF THE WEIGHTED q -GENOCCHI NUMBERS AND POLYNOMIALS

For $\alpha \in \mathbb{N} \cup \{0\}$, the weighted q -Genocchi polynomials are defined by means of the following generating function:

For $x \in \mathbb{C}$,

$$(2.1) \quad \sum_{n=0}^{\infty} \tilde{G}_{n,q}(x | \alpha) \frac{t^n}{n!} = [2 : q] t \sum_{n=0}^{\infty} (-1)^n q^n e^{t[n+x;q^\alpha]}.$$

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As a special case $x = 0$ into (2.1), $\tilde{G}_{n,q}(0 \mid \alpha) := \tilde{G}_{n,q}(\alpha)$ are called weighted q -Genocchi numbers. By (2.1), we readily derive the following

$$(2.2) \quad \frac{\tilde{G}_{n+1,q}(x \mid \alpha)}{n+1} = \frac{[2 : q]}{[\alpha : q]^n (1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{1 + q^{\alpha l + 1}},$$

where $\binom{n}{l}$ is the binomial coefficient. By expression (2.1), we see that

$$(2.3) \quad \tilde{G}_{n,q}(x \mid \alpha) = q^{-\alpha x} \left(q^{\alpha x} \tilde{G}_q(\alpha) + [x : q^\alpha] \right)^n,$$

with the usual convention of replacing $\left(\tilde{G}_q(\alpha) \right)^n$ by $\tilde{G}_{n,q}(\alpha)$ is used (for details, see [7], [8]).

Let $\tilde{T}_q^{(\alpha)}(x, t)$ be the generating function of weighted q -Genocchi polynomials as follows:

$$(2.4) \quad \tilde{T}_q^{(\alpha)}(x, t) = \sum_{n=0}^{\infty} \tilde{G}_{n,q}(x \mid \alpha) \frac{t^n}{n!}.$$

Then, we easily notice that

$$(2.5) \quad \tilde{T}_q^{(\alpha)}(x, t) = [2 : q] t \sum_{n=0}^{\infty} (-1)^n q^n e^{t[n+x:q^\alpha]}.$$

From expressions (2.4) and (2.5), we procure the followings:

For k (=even) and $n, \alpha \in \mathbb{N} \cup \{0\}$, we have

$$(2.6) \quad q^k \frac{\tilde{G}_{n+1,q}(k \mid \alpha)}{n+1} - \frac{\tilde{G}_{n+1,q}(\alpha)}{n+1} = [2 : q] \sum_{l=0}^{k-1} (-1)^l q^{k-l-1} [l : q^\alpha]^n.$$

For k (=odd) and $n, \alpha \in \mathbb{N} \cup \{0\}$, we have

$$(2.7) \quad q^k \frac{\tilde{G}_{n+1,q}(k \mid \alpha)}{n+1} + \frac{\tilde{G}_{n+1,q}(\alpha)}{n+1} = [2 : q] \sum_{l=0}^{k-1} (-1)^l q^{k-l-1} [l : q^\alpha]^n.$$

Via Eq. (2.5), we easily obtain the following:

$$(2.8) \quad \tilde{G}_{n,q}(x \mid \alpha) = q^{-\alpha x} \sum_{k=0}^n \binom{n}{k} q^{\alpha k x} \tilde{G}_{k,q}(\alpha) [x : q^\alpha]^{n-k}.$$

From (2.6)-(2.8), we get the following:

$$(2.9) \quad \begin{aligned} & [2 : q] \sum_{l=0}^{k-1} (-1)^l q^{k-l-1} [l : q^\alpha]^n \\ &= (q^{\alpha k n} - 1) \frac{\tilde{G}_{n+1,q}(\alpha)}{n+1} + q^{-\alpha k} \sum_{j=0}^n \frac{1}{n+1} \binom{n+1}{j} q^{\alpha j k} \tilde{G}_{k,q}(\alpha) [k : q^\alpha]^{n+1-k}, \end{aligned}$$

here k is an even positive integer. If k is an odd positive integer. Then, we can derive the following equality:

$$(2.10) \quad [2 : q] \sum_{l=0}^{k-1} (-1)^l q^{k-l-1} [l : q^\alpha]^n \\ = (q^{\alpha k n} + 1) \frac{\tilde{G}_{n+1,q}(\alpha)}{n+1} + q^{-\alpha k} \sum_{j=0}^n \frac{1}{n+1} \binom{n+1}{j} q^{\alpha j k} \tilde{G}_{k,q}(\alpha) [k : q^\alpha]^{n+1-k}.$$

3. WEIGHTED q -GENOCCHI-ZETA FUNCTION

The famous Genocchi polynomials were defined as

$$(3.1) \quad \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi \text{ cf. [4].}$$

For $s \in \mathbb{C}$, $x \in \mathbb{R}$ with $0 \leq x < 1$, Genocchi-Zeta function are given by

$$(3.2) \quad \zeta_G(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s},$$

and

$$(3.3) \quad \zeta_G(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}.$$

By (3.1), (3.2) and (3.3), Genocchi-Zeta functions are related to the Genocchi numbers as follows:

$$\zeta_G(-n) = \frac{G_{n+1}}{n+1}.$$

Moreover, it is simple to see

$$\zeta_G(-n, x) = \frac{G_{n+1}(x)}{n+1}.$$

The weighted q -Genocchi Hurwitz-Zeta type function are defined by

$$\tilde{\zeta}_{G,q}(s, x | \alpha) = [2 : q] \sum_{m=0}^{\infty} \frac{(-1)^m q^m}{[m+x : q^\alpha]^s}.$$

Similarly, weighted q -Genocchi-Zeta function are given by

$$\tilde{\zeta}_{G,q}(s | \alpha) = [2 : q] \sum_{m=1}^{\infty} \frac{(-1)^m q^m}{[m : q^\alpha]^s}.$$

For $n, \alpha \in \mathbb{N} \cup \{0\}$, we have

$$\tilde{\zeta}_{G,q}(-n | \alpha) = \frac{\tilde{G}_{n+1,q}(\alpha)}{n+1}.$$

We now consider the function $\tilde{G}_q(n : \alpha)$ as the analytic continuation of weighted q -Genocchi numbers. All the weighted q -Genocchi numbers agree with $\tilde{G}_q(n : \alpha)$, the analytic continuation of weighted q -Genocchi numbers evaluated at n . For $n \geq 0$, $\tilde{G}_q(n : \alpha) = \tilde{G}_{n,q}(\alpha)$.

We can now state $\tilde{G}_q(s : \alpha)$ in terms of $\tilde{\zeta}_{G,q}(s | \alpha)$, the derivative of $\tilde{\zeta}_{G,q}(s : \alpha)$

$$\frac{\tilde{G}_q(s+1 : \alpha)}{s+1} = \tilde{\zeta}_{G,q}(-s | \alpha), \quad \frac{\tilde{G}_q(s+1 : \alpha)}{s+1} = \tilde{\zeta}_{G,q}(-s | \alpha).$$

For $n, \alpha \in \mathbb{N} \cup \{0\}$

$$\frac{\tilde{G}_q(2n+1 : \alpha)}{2n+1} = \tilde{\zeta}_{G,q}(-2n | \alpha).$$

This is suitable for the differential of the functional equation and so supports the coherence of $\tilde{G}_q(s : \alpha)$ and $\tilde{G}_q(s : \alpha)$ with $\tilde{G}_{n,q}(\alpha)$ and $\tilde{\zeta}_{G,q}(s | \alpha)$. From the analytic continuation of weighted q -Genocchi numbers, we derive as follows:

$$\frac{\tilde{G}_q(s+1 : \alpha)}{s+1} = \tilde{\zeta}_{G,q}(-s | \alpha) \quad \text{and} \quad \frac{\tilde{G}_q(-s+1 : \alpha)}{-s+1} = \tilde{\zeta}_{G,q}(s | \alpha).$$

Moreover, we derive the following:

For $n \in \mathbb{N} - \{1\}$

$$\frac{\tilde{G}_{-n+1,q}(\alpha)}{-n+1} = \frac{\tilde{G}_q(-n+1 : \alpha)}{-n+1} = \tilde{\zeta}_{G,q}(n | \alpha).$$

The curve $\tilde{G}_q(s : a)$ review quickly the points $\tilde{G}_{-s,q}(\alpha)$ and grows $\sim n$ asymptotically $(-n) \rightarrow -\infty$. The curve $\tilde{G}_q(s : a)$ review quickly the point $\tilde{G}_q(-s : a)$. Then, we procure the following:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\tilde{G}_q(-n+1 : \alpha)}{-n+1} &= \lim_{n \rightarrow \infty} \tilde{\zeta}_{G,q}(n | \alpha) = \lim_{n \rightarrow \infty} \left([2 : q] \sum_{m=1}^{\infty} \frac{(-1)^m q^m}{[m : q^\alpha]^n} \right) \\ &= \lim_{n \rightarrow \infty} \left(-q[2 : q] + [2 : q] \sum_{m=2}^{\infty} \frac{(-1)^m q^m}{[m : q^\alpha]^n} \right) = -q^2 [2 : q^{-1}]. \end{aligned}$$

From this, we easily note that

$$\frac{\tilde{G}_q(-n+1 : \alpha)}{-n+1} = \tilde{\zeta}_{G,q}(n | \alpha) \mapsto \frac{\tilde{G}_q(-s+1 : \alpha)}{-s+1} = \tilde{\zeta}_{G,q}(s | \alpha).$$

4. ANALYTIC CONTINUATION OF WEIGHTED q -GENOCCHI POLYNOMIALS

For coherence with the redefinition of $\tilde{G}_{n,q}(\alpha) = \tilde{G}_q(n : \alpha)$, we have

$$\tilde{G}_{n,q}(x | \alpha) = q^{-\alpha x} \sum_{k=0}^n \binom{n}{k} q^{\alpha k x} \tilde{G}_{k,q}(\alpha) [x : q^\alpha]^{n-k}.$$

Let $\Gamma(s)$ be Euler-gamma function. Then the analytic continuation can be get as

$$\begin{aligned}
n &\mapsto s \in \mathbb{R}, x \mapsto w \in \mathbb{C}, \\
\tilde{G}_{n,q}(\alpha) &\mapsto \tilde{G}_q(k+s-[s]:\alpha) = \tilde{\zeta}_{G,q}(-(k+s-[s])|\alpha), \\
\binom{n}{k} &= \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} \mapsto \frac{\Gamma(s+1)}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)} \\
\tilde{G}_{s,q}(w|\alpha) &\mapsto \tilde{G}_q(s,w:\alpha) = q^{-\alpha w} \sum_{k=-1}^{[s]} \frac{\Gamma(s+1)\tilde{G}_q(k+(s-[s]):\alpha)q^{\alpha w(k+(s-[s]))}}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)} [w:q^\alpha]^{[s]-k} \\
&= q^{-\alpha w} \sum_{k=0}^{[s]+1} \frac{\Gamma(s+1)\tilde{G}_q(-1+k+(s-[s]):\alpha)q^{\alpha w(k-1+(s-[s]))}}{\Gamma(k+(s-[s]))\Gamma(2+[s]-k)} [w:q^\alpha]^{[s]+1-k}.
\end{aligned}$$

Here $[s]$ gives the integer part of s , and so $s-[s]$ gives the fractional part.

Deformation of the curve $\tilde{G}_q(1,w:\alpha)$ into the curve of $\tilde{G}_q(2,w:\alpha)$ is by means of the real analytic cotinuation $\tilde{G}_q(s,w:\alpha)$, $1 \leq s \leq 2$, $-0.5 \leq w \leq 0.5$.

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